

A GENERALISATION OF CERONE'S IDENTITY AND APPLICATIONS

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ABSTRACT. An identity due to P. Cerone for the Čebyšev functional is extended for Stieltjes integrals. A sharp inequality and its application in approximating Stieltjes integrals are also given.

1. INTRODUCTION

In 2001, P. Cerone [1] established the following identity for the Čebyšev functional:

$$(1.1) \quad T(f, g) := \frac{1}{b-a} \int_a^b f(t) g(t) dt - \frac{1}{b-a} \int_a^b f(t) dt \cdot \frac{1}{b-a} \int_a^b g(t) dt$$

$$= \frac{1}{(b-a)^2} \int_a^b \left[(t-a) \int_t^b g(s) ds - (b-t) \int_a^t g(s) ds \right] df(t),$$

provided f is of bounded variation on $[a, b]$ and g is continuous on $[a, b]$. He proved (1.1) on utilising the auxiliary function $\Psi : [a, b] \rightarrow \mathbb{R}$,

$$(1.2) \quad \Psi(t) := (t-a) \int_t^b g(s) ds - (b-t) \int_a^t g(s) ds$$

and integrating by parts in the Stieltjes integral $\int_a^b \Psi(t) df(t)$, which exists, since f is of bounded variation and Ψ is differentiable on (a, b) .

One may observe that the result remains valid if one assumes that g is Lebesgue integrable on $[a, b]$ and f is of bounded variation. This follows by the fact that, in this case Ψ becomes absolutely continuous on $[a, b]$, the Stieltjes integral $\int_a^b \Psi(t) df(t)$ still exists and the argument will follow as in [1].

The weighted version of this inequality has been obtained in the same paper [1] and can be stated as:

$$(1.3) \quad T(f, g; p) := \frac{1}{\int_a^b p(s) ds} \int_a^b p(t) f(t) g(t) dt$$

$$- \frac{1}{\int_a^b p(s) ds} \int_a^b p(t) f(t) dt \cdot \frac{1}{\int_a^b p(s) ds} \int_a^b p(t) g(t) dt$$

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$$= \frac{1}{\left(\int_a^b p(s) ds\right)^2} \int_a^b \left[\int_a^t p(s) ds \int_t^b p(s) g(s) ds - \int_t^b p(s) ds \int_a^t p(s) g(s) ds \right] df(t),$$

provided f is of bounded variation on $[a, b]$ and p, g are continuous on $[a, b]$ with $\int_a^b p(s) ds > 0$. The same remark for the extension of the identity in the case that p, g are Lebesgue integrable on $[a, b]$ so that pg is also integrable, may apply.

The above two identities have been applied in [1] to obtain some interesting new bounds for the Čebyšev functionals $T(f, g)$ and $T(f, g; p)$ from which we only mention the following:

$$(1.4) \quad |T(f, g)| \leq \frac{1}{(b-a)^2} \times \begin{cases} \sup_{t \in [a, b]} |\Psi(t)| V_a^b(f); \\ L \int_a^b |\Psi(t)| dt & \text{for } f \text{ } L\text{-Lipschitzian;} \\ \int_a^b |\Psi(t)| df(t) & \text{for } f \text{ monotonic nondecreasing,} \end{cases}$$

where $V_a^b(f)$ is the total variation of f on $[a, b]$, $\Psi(t)$ is given by (1.2), and

$$(1.5) \quad |T(f, g; p)| \leq \frac{1}{\left(\int_a^b p(s) ds\right)^2} \times \begin{cases} \sup_{t \in [a, b]} |\Psi_p(t)| V_a^b(f); \\ L \int_a^b |\Psi_p(t)| dt & \text{if } f \text{ is } L\text{-Lipschitzian;} \\ \int_a^b |\Psi_p(t)| df(t) & \text{for } f \text{ monotonically nondecreasing,} \end{cases}$$

where in this case the wighted auxiliary mapping Ψ_p is defined as $\Psi_p : [a, b] \rightarrow \mathbb{R}$,

$$\Psi_p(t) := \int_a^t p(s) ds \int_t^b p(s) g(s) ds - \int_t^b p(s) ds \int_a^t p(s) g(s) ds.$$

For other inequalities and applications for moments, see [1].

For further results, see the follow up paper [2] where various lower and other upper bounds were established.

2. A RELATED FUNCTIONAL

In [4], the authors have considered the following functional

$$(2.1) \quad D(f; u) := \int_a^b f(x) du(x) - [u(b) - u(a)] \cdot \frac{1}{b-a} \int_a^b f(t) dt,$$

provided that the Stieltjes integral $\int_a^b f(x) du(x)$ exists.

This functional plays an important role in approximating the Stieltjes integral $\int_a^b f(x) du(x)$ in terms of the Riemann integral $\int_a^b f(t) dt$ and the divided difference of the integrator u . Therefore, further bounds on $D(f; u)$ will generate a flow of different error estimates for the approximation of the Stieltjes integral that plays an

important role in various fields of Analysis, Numerical Analysis, Integral Operator Theory, Probability & Statistics and other fields of Modern Mathematics.

In [4], the following result in estimating the above functional $D(f; u)$ has been obtained:

$$(2.2) \quad |D(f; u)| \leq \frac{1}{2} L (M - m) (b - a),$$

provided u is L -Lipschitzian and f is Riemann integrable and with the property that there exists the constants $m, M \in \mathbb{R}$ such that

$$(2.3) \quad m \leq f(x) \leq M \quad \text{for any } x \in [a, b].$$

The constant $\frac{1}{2}$ is best possible in (2.2) in the sense that it cannot be replaced by a smaller quantity.

If one assumes that u is of bounded variation and f is K -Lipschitzian, then $D(f, u)$ satisfies the inequality [5]

$$(2.4) \quad |D(f; u)| \leq \frac{1}{2} K (b - a) \bigvee_a^b(u).$$

Here the constant $\frac{1}{2}$ is also best possible.

The above inequalities have been used in [4] and [5] for obtaining inequalities between special means and on estimating the error in approximating the Stieltjes integral $\int_a^b f(x) du(x)$ in terms of the Riemann integral for the function f and the divided difference of u .

Now, for the function $u : [a, b] \rightarrow \mathbb{R}$, consider the following auxiliary mappings Φ, Γ and Δ [3]:

$$\begin{aligned} \Phi(t) &:= \frac{(t-a)u(b) + (b-t)u(a)}{b-a} - u(t), \quad t \in [a, b], \\ \Gamma(t) &:= (t-a)[u(b) - u(t)] - (b-t)[u(t) - u(a)], \quad t \in [a, b], \\ \Delta(t) &:= [u; b, t] - [u; t, a], \quad t \in (a, b), \end{aligned}$$

where $[u; \alpha, \beta]$ is the divided difference of u in α, β , i.e.,

$$[u; \alpha, \beta] := \frac{u(\alpha) - u(\beta)}{\alpha - \beta}.$$

The following representation of $D(f, u)$ may be stated.

Theorem 1. *Let $f, u : [a, b] \rightarrow \mathbb{R}$ be such that the Stieltjes integral $\int_a^b f(t) du(t)$ and the Riemann integral $\int_a^b f(t) dt$ exist. Then*

$$\begin{aligned} (2.5) \quad D(f, u) &= \int_a^b \Phi(t) df(t) = \frac{1}{b-a} \int_a^b \Gamma(t) df(t) \\ &= \frac{1}{b-a} \int_a^b (t-a)(b-t) \Delta(t) df(t). \end{aligned}$$

Proof. Since $\int_a^b f(t) du(t)$ exists, hence $\int_a^b \Phi(t) df(t)$ also exists, and the integration by parts formula for Stieltjes integrals gives that

$$\begin{aligned} \int_a^b \Phi(t) df(t) &= \int_a^b \left[\frac{(t-a)u(b) + (b-t)u(a)}{b-a} - u(t) \right] df(t) \\ &= \left[\frac{(t-a)u(b) + (b-t)u(a)}{b-a} - u(t) \right] f(t) \Big|_a^b \\ &\quad - \int_a^b f(t) d \left[\frac{(t-a)u(b) + (b-t)u(a)}{b-a} - u(t) \right] \\ &= - \int_a^b f(t) \left[\frac{u(b) - u(a)}{b-a} dt - du(t) \right] = D(f, u), \end{aligned}$$

proving the required identity. ■

Remark 1. The identity (2.5) has been established in [3]. There were some typographical errors in [3] that have been corrected above.

Remark 2. If u is an integral, i.e., $u(t) = \int_a^t g(s) ds$, $t \in [a, b]$, then

$$\begin{aligned} \Phi(t) &= \frac{t-a}{b-a} \int_a^b g(s) ds - \int_a^t g(s) ds, \\ \Gamma(t) &= (t-a) \int_t^b g(s) ds - (b-t) \int_a^t g(s) ds, \\ \Delta(t) &= \frac{\int_t^b g(s) ds}{b-t} - \frac{\int_a^t g(s) ds}{t-a}, \end{aligned}$$

and then, from (2.5), one may recapture Cerone's identity (1.1) for the Čebyšev functional $T(f, g)$.

Since it well known that u is an integral if and only if u is absolutely continuous, and in this case $g(s) = u'(s)$ for $s \in [a, b]$, hence (2.5) is indeed a proper generalisation of (1.1) holding for larger classes of functions than the absolutely continuous functions.

Remark 3. If one chooses $u : [a, b] \rightarrow \mathbb{R}$,

$$u(t) = \frac{\int_a^t p(s) g(s) ds}{\int_a^b p(s) ds}, \quad t \in [a, b],$$

where p, g are Lebesgue integrable with pg is also integrable and $\int_a^b p(s) ds \neq 0$, then the identity (2.5) produces the representation:

$$\begin{aligned} (2.6) \quad E(f, g; p) &:= \frac{\int_a^b p(s) f(s) g(s) ds}{\int_a^b p(s) ds} - \frac{\int_a^b p(s) g(s) ds}{\int_a^b p(s) ds} \cdot \frac{1}{b-a} \int_a^b f(t) dt \\ &= \int_a^b \Phi_p(t) df(t) = \frac{1}{b-a} \int_a^b \Gamma_p(t) df(t) \\ &= \frac{1}{b-a} \int_a^b (t-a)(b-t) \Delta_p(t) df(t), \end{aligned}$$

where

$$\begin{aligned}\Phi_p(t) &:= \frac{t-a}{b-a} \cdot \frac{\int_a^b p(s) g(s) ds}{\int_a^b p(s) ds} - \frac{\int_a^t p(s) g(s) ds}{\int_a^b p(s) ds}, \\ \Gamma_p(t) &:= (t-a) \frac{\int_t^b p(s) g(s) ds}{\int_a^b p(s) ds} - (b-t) \frac{\int_a^t p(s) g(s) ds}{\int_a^b p(s) ds}\end{aligned}$$

and

$$\Delta_p(t) := \frac{\int_t^b p(s) g(s) ds}{(b-t) \int_a^b p(s) ds} - \frac{\int_a^t p(s) g(s) ds}{(t-a) \int_a^b p(s) ds}.$$

One must observe that the identity (2.6) is not the same as Cerone's identity for weighted integrals (1.3).

For recent inequalities related to $D(f; u)$ for various pairs of functions (f, u) , see [3, pp. 112-118].

3. A BOUND FOR f OF BOUNDED VARIATION AND u CONTINUOUS

It is known that if u is continuous on $[a, b]$ and f is of bounded variation on $[a, b]$, then the Stieltjes integral $\int_a^b f(t) du(t)$ exists. This integral may exist even for larger classes of integrators f , for instance, piecewise continuous functions for which the discontinuities of the integrand f do not overlap with those of the integrator u .

The following result may be stated:

Theorem 2. *Let $f : [a, b] \rightarrow \mathbb{R}$ be of bounded variation on $[a, b]$ and $u : [a, b] \rightarrow \mathbb{R}$ such that there exist the constants $\gamma, \Gamma \in \mathbb{R}$ with:*

$$(3.1) \quad \gamma \leq u(t) \leq \Gamma \quad \text{for any } t \in [a, b]$$

and the Stieltjes integral $\int_a^b f(t) du(t)$ exists. Then

$$(3.2) \quad |D(f; u)| \leq (\Gamma - \gamma) \bigvee_a^b(f).$$

The multiplicative constant 1 in front of $\Gamma - \gamma$ cannot be replaced by a smaller quantity.

Proof. By (1.1), we obviously have:

$$\begin{aligned}\gamma(b-t) &\leq (b-t)u(a) \leq (b-t)\Gamma, \\ \gamma(t-a) &\leq (t-a)u(b) \leq (t-a)\Gamma, \\ -(b-a)\Gamma &\leq -(b-a)u(t) \leq -(b-a)\gamma,\end{aligned}$$

which gives by addition and division with $b-a$ that

$$-(\Gamma - \gamma) \leq \frac{(b-t)u(a) + (t-a)u(b)}{b-a} - u(t) \leq \Gamma - \gamma,$$

showing that $|\Phi(t)| \leq \Gamma - \gamma$ for any $t \in [a, b]$.

Taking into account that for φ bounded and ψ of bounded variation on $[a, b]$ one has

$$\left| \int_a^b \varphi(t) d\psi(t) \right| \leq \sup_{t \in [a, b]} |\varphi(t)| \bigvee_a^b(\psi),$$

provided the Stieltjes integral exists, we have by (2.5) that

$$|D(f; u)| \leq \sup_{t \in [a, b]} |\phi(t)| \bigvee_a^b(f) \leq (\Gamma - \gamma) \bigvee_a^b(f),$$

proving the required inequality (3.2).

Now, for the sharpness of the inequality.

Assume that there exists a $c > 0$ such that

$$(3.3) \quad |D(f; u)| \leq c(\Gamma - \gamma) \bigvee_a^b(f),$$

where u and f are as in the hypothesis of the theorem.

Consider $u, f : [a, b] \rightarrow \mathbb{R}$ with

$$u(t) = \frac{1}{2} \left(t - \frac{a+b}{2} \right)^2, \quad f(t) = \operatorname{sgn} \left(t - \frac{a+b}{2} \right), \quad t \in [a, b].$$

Then u is continuous, f is of bounded variation, the integral $\int_a^b f(t) du(t)$ exists and

$$\begin{aligned} \bigvee_a^b(f) &= 2, \quad \int_a^b f(t) dt = 0, \\ \Gamma &= \sup_{t \in [a, b]} u(t) = \frac{(b-a)^2}{8}, \quad \gamma = \inf_{t \in [a, b]} u(t) = 0, \\ \int_a^b f(t) du(t) &= \int_a^b \operatorname{sgn} \left(t - \frac{a+b}{2} \right) \left(t - \frac{a+b}{2} \right) dt \\ &= \int_a^b \left| t - \frac{a+b}{2} \right| dt = \frac{(b-a)^2}{4}. \end{aligned}$$

Substituting into (3.3) we get $\frac{(b-a)^2}{4} \leq \frac{c(b-a)^2}{4}$, which implies that $c \geq 1$. ■

Corollary 1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be of bounded variation and $u : [a, b] \rightarrow \mathbb{R}$ continuous on $[a, b]$. Then:*

$$(3.4) \quad |D(f; u)| \leq \left[\max_{t \in [a, b]} u(t) - \min_{t \in [a, b]} u(t) \right] \bigvee_a^b(f).$$

The inequality (3.4) is sharp.

If we consider the Čebyšev functional $T(f, g)$, then we can state the following corollary as well:

Corollary 2. *Let $f : [a, b] \rightarrow \mathbb{R}$ be of bounded variation and $g : [a, b] \rightarrow \mathbb{R}$ a Lebesgue integrable function such that there exists the constants m and M with*

$$(3.5) \quad m \leq g(s) \leq M \quad \text{for a.e. } s \in [a, b].$$

Then

$$(3.6) \quad |T(f, g)| \leq (b-a)(M-m) \bigvee_a^b(f).$$

Proof. We choose $u(t) := \int_a^t g(s) ds$ which is continuous on $[a, b]$ and satisfies the inequality (3.1) with $\gamma = (b-a)m$ and $\Gamma = (b-a)M$ and apply Theorem 2. ■

Remark 4. If we assume that for the Lebesgue integrable function g , $\int_a^b g(s) ds$ satisfies the condition

$$\gamma \leq \int_a^t g(s) ds \leq \Gamma \quad \text{for any } t \in [a, b],$$

then

$$|T(f, g)| \leq (\Gamma - \gamma) \bigvee_a^b(f)$$

and the inequality is sharp. The equality case is realised for $g(t) = t - \frac{a+b}{2}$ and $f(t) = \operatorname{sgn}\left(t - \frac{a+b}{2}\right)$, $t \in [a, b]$.

It is an open problem whether or not the bound in (3.6) is sharp.

Remark 5. If $p, g \in L[a, b]$ so that $pg \in L[a, b]$ and $\int_a^b p(s) ds \neq 0$ and there exists the constants δ, Δ so that

$$\delta \leq \frac{\int_a^t p(s) g(s) ds}{\int_a^b p(s) ds} \leq \Delta$$

for any $t \in [a, b]$, then

$$|E(f, g; p)| \leq (\Delta - \delta) \bigvee_a^b(f).$$

The last inequality is sharp.

4. APPLICATION FOR APPROXIMATING THE STIELTJES INTEGRAL

Let us consider the partition of the interval $[a, b]$ given by

$$I_n : a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b.$$

Denote $v(I_n) := \max\{h_i | i = 0, \dots, n-1\}$, where $h_i := t_{i+1} - t_i$, $i = 0, \dots, n-1$. If $u : [a, b] \rightarrow \mathbb{R}$ is continuous on $[a, b]$ and if we define

$$M_i := \sup_{t \in [t_i, t_{i+1}]} u(t), \quad m_i := \inf_{t \in [t_i, t_{i+1}]} u(t)$$

and

$$v(u, I_n) := \max_{0 \leq i \leq n-1} (M_i - m_i),$$

then, obviously, by the continuity of u on $[a, b]$, for any $\varepsilon \geq 0$, there exists a $\delta > 0$ and a division I_n with norm $v(I_n) < \delta$ such that $v(u, I_n) < \varepsilon$.

Consider now the quadrature rule

$$(4.1) \quad S_n(f, u, I_n) := \sum_{i=0}^{n-1} \frac{[u(t_{i+1}) - u(t_i)]}{t_{i+1} - t_i} \cdot \int_{t_i}^{t_{i+1}} f(t) dt,$$

provided u is continuous on $[a, b]$ and f is of bounded variation on $[a, b]$.

We may state the following result in approximating the Stieltjes integral:

Theorem 3. Let $f, u : [a, b] \rightarrow \mathbb{R}$ be such that f is of bounded variation on $[a, b]$ and u is continuous on $[a, b]$. Then for any division I_n as above,

$$(4.2) \quad \int_a^b f(t) du(t) = S_n(f, u, I_n) + R_n(f, u, I_n),$$

where the remainder $R_n(f, u, I_n)$ satisfies the estimate:

$$(4.3) \quad |R_n(f, u, I_n)| \leq v(u, I_n) \bigvee_a^b(f).$$

Proof. Applying Theorem 2 on the intervals $[t_i, t_{i+1}]$, $i = 0, \dots, n-1$, we have successively:

$$\begin{aligned} |R_n(f, u, I_n)| &= \left| \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} f(t) du(t) - \frac{u(t_{i+1}) - u(t_i)}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} f(t) dt \right| \\ &\leq \sum_{i=0}^{n-1} \left| \int_{t_i}^{t_{i+1}} f(t) du(t) - \frac{u(t_{i+1}) - u(t_i)}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} f(t) dt \right| \\ &\leq \sum_{i=0}^{n-1} (M_i - m_i) \bigvee_{t_i}^{t_{i+1}}(f) \leq v(u, I_n) \bigvee_a^b(f) \end{aligned}$$

and the estimate (4.3) is obtained. ■

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